Notes on dark energy and the cosmic horizon

Nathan Reed, December 2011

The Friedmann equation for a universe that includes only dark energy (negligible matter and radiation) is

$$\frac{d^2a}{dt^2} = -\tfrac{4}{3}\pi G\rho_{DE0}(3w+1)a^{-3w-2}, \tag{1}$$

where a is the universal scale factor (a ratio of the universe's size vs the present day, i.e. $a(0) = 1$ if $t = 0$ is the present); G is the gravitational constant, ρ_{DE0} is the present-day density of dark energy, equal to 6.78×10^{-27} kg/m^3, and w is the state parameter of dark energy.

Define $S = \frac{2}{3}\pi G\rho_{DE0} = 9.44 \times 10^{-22}$ yr^{-2}, as a convenience, since this factor shows up repeatedly in the following equations. Then the Friedmann equation is

$$\frac{d^2a}{dt^2} = -2S(3w+1)a^{-3w-2}. \tag{2}$$

The goal is to first solve this equation to get $a(t)$, then to evaluate the distance to the cosmic horizon, defined by

$$d_e(t) \equiv a(t)\, c \int_t^\infty \frac{dt'}{a(t')}. \tag{3}$$

Now (2) is an *autonomous* differential equation, meaning that the right side depends only on a itself, and not directly on t. This means we cannot just integrate it twice to solve it; instead, we must guess the *form* of the solution and then solve for whatever constants appear in that form. Equation (2) splits into two cases based on the power to which a is raised on the right-hand side. When $w = -1$, this power is 1, so that the equation is saying a's derivative is proportional to a itself; this suggests an exponential form for the solution. For all other values of w we have that a's derivative is proportional to a power of a, suggesting a power-law form for the solution. We'll solve each of these cases separately.

Moreover, (2) being autonomous also means that it is insensitive to a global time translation. We'll make use of this freedom to shift all solutions so that $a(0) = 1$, i.e. so that $t = 0$ represents the present day.

Case 1: $w = -1$

This is the easy case. Equation (2) reduces to

$$\frac{d^2a}{dt^2} = 4Sa,$$

so if $a = e^{kt}$, this equation will be satisfied when $k^2 = 4S$. Therefore,

$$a(t) = e^{2\sqrt{S}\,t}. \tag{4}$$

Note that $a(0) = 1$ already, so no time translation is required. We can proceed straight to finding the horizon distance from (3):

$$\begin{aligned} d_e(t) &= ce^{2\sqrt{S}\,t} \int_t^\infty e^{-2\sqrt{S}\,t'}\, dt' \\ &= \frac{-c}{2\sqrt{S}}\, e^{2\sqrt{S}\,t} \left[e^{-2\sqrt{S}\,t'} \right]_t^\infty \\ &= \frac{c}{2\sqrt{S}}\, e^{2\sqrt{S}\,t}\, e^{-2\sqrt{S}\,t} \\ &= \frac{c}{2\sqrt{S}}. \end{aligned} \tag{5}$$

Case 2: $w \neq -1$

As mentioned earlier, (2) suggests a power-law form for this case, so we'll use the trial solution

$$a(t) = C(t - t_{\text{ref}})^\beta, \tag{6}$$

where C, β, t_{ref} are constants to be determined. C and β must be chosen to satisfy (2), and then t_{ref} can be chosen to shift the solution in time to make $a(0) = 1$.

Differentiating (6),

$$\begin{aligned} \frac{da}{dt} &= C\beta(t - t_{\text{ref}})^{\beta-1}, \\ \frac{da^2}{dt^2} &= C\beta(\beta-1)(t - t_{\text{ref}})^{\beta-2} \\ &= C\beta(\beta-1)\left[(t - t_{\text{ref}})^\beta\right]^{\frac{\beta-2}{\beta}} \\ &= C^{1-\frac{\beta-2}{\beta}}\beta(\beta-1)\left[C(t - t_{\text{ref}})^\beta\right]^{\frac{\beta-2}{\beta}} \\ &= C^{1-\frac{\beta-2}{\beta}}\beta(\beta-1)a^{\frac{\beta-2}{\beta}}. \end{aligned} \tag{7}$$

The power to which a is raised on the right-hand side here must match that in (2), which determines β:

$$\begin{aligned} \frac{\beta-2}{\beta} &= -3w - 2 \\ \beta &= \frac{2}{3w+3}. \end{aligned} \tag{8}$$

Similarly, the constant factor on the right-hand side of (7) must match that of (2):

$$\begin{aligned}
-2S(3w+1) &= C^{1-\frac{\beta-2}{\beta}}\beta(\beta-1) \\
&= C^{3w+2}\frac{2}{3w+3}\left(\frac{2}{3w+3}-1\right) \\
&= C^{3w+2}\frac{2}{3w+3}\;\frac{-3w-1}{3w+3} \\
S &= C^{3w+2}\frac{1}{(3w+3)^2} \\
C &= \left[(3w+3)^2 S\right]^{\frac{1}{3w+3}}.
\end{aligned} \tag{9}$$

Putting (6), (8), and (9) together gives

$$a(t) = \left[(3w+3)^2(t-t_{\mathrm{ref}})^2 S\right]^{\frac{1}{3w+3}}. \tag{10}$$

Now we can apply the constraint $a(0) = 1$ to solve for t_{ref}:

$$\begin{aligned}
1 &= \left[(3w+3)^2 t_{\mathrm{ref}}^2 S\right]^{\frac{1}{3w+3}} \\
t_{\mathrm{ref}} &= \pm\frac{1}{(3w+3)\sqrt{S}}
\end{aligned} \tag{11}$$

At this point we need to choose the sign for t_{ref}. It turns out that the negative sign is the correct one here; if the positive sign is taken, we get a *time-reversed* solution in which the universe is shrinking rather than expanding. (This occurs because (2), being a second-order differential equation, is insensitive to which way time is going! Reversing the direction of time negates da/dt, but leaves da^2/dt^2 unchanged.)

Combining (10) and (11), with the negative sign on t_{ref}, gives

$$\begin{aligned}
a(t) &= \left[(3w+3)^2\left(t+\frac{1}{(3w+3)\sqrt{S}}\right)^2 S\right]^{\frac{1}{3w+3}} \\
&= \left[1+(3w+3)\sqrt{S}\,t\right]^{\frac{2}{3w+3}}.
\end{aligned} \tag{12}$$

It's worth noting now that although the solution has the same form for $w < -1$ and $w > -1$, its behavior is different in those cases, because $3w+3$ is positive when $w > -1$ and negative when $w < -1$. This means that for $w > -1$, the solution is an increasing value raised to a positive power, so it grows slower than the exponential (4). But when $w < -1$, the solution is a *decreasing* value raised to a *negative* power—which grows *hyperexponentially* and indeed reaches a singularity when the quantity in brackets goes to zero, at the finite time

$$t_{\mathrm{rip}} = \frac{1}{|3w+3|\sqrt{S}}. \tag{13}$$

This is the date of the Big Rip—the self-destruction of space-time.

We can now proceed to calculate the distance to the cosmic horizon by using (12) in (3):

$$d_e = c\left[1+(3w+3)\sqrt{S}\,t\right]^{\frac{2}{3w+3}} \int_t^\infty \left[1+(3w+3)\sqrt{S}\,t'\right]^{\frac{-2}{3w+3}} dt'$$

Letting $u = 1+(3w+3)\sqrt{S}\,t'$ and changing variables,

$$\begin{aligned} d_e &= \frac{c}{(3w+3)\sqrt{S}}\left[1+(3w+3)\sqrt{S}\,t\right]^{\frac{2}{3w+3}} \int_{1+(3w+3)\sqrt{S}\,t}^\infty u^{\frac{-2}{3w+3}}\,du \\ &= \frac{c}{(3w+3)\sqrt{S}}\left(1-\frac{2}{3w+3}\right)^{-1}\left[1+(3w+3)\sqrt{S}\,t\right]^{\frac{2}{3w+3}} u^{1-\frac{2}{3w+3}}\Big|_{1+(3w+3)\sqrt{S}\,t}^\infty \qquad (14) \\ &= \frac{c}{(3w+1)\sqrt{S}}\left[1+(3w+3)\sqrt{S}\,t\right]^{\frac{2}{3w+3}} u^{1-\frac{2}{3w+3}}\Big|_{1+(3w+3)\sqrt{S}\,t}^\infty \end{aligned}$$

Now for $w > -1$, the power $1-\frac{2}{3w+3}$ is negative, so the upper bound of the integration vanishes. The factor $3w+1$ in the denominator is also negative, so to make all the factors positive we'll put an absolute value around it, which also gets rid of the negative sign from the lower bound of the integration.

$$d_e = \frac{c}{|3w+1|}\left[S^{-1/2} + (3w+3)t\right]. \qquad (15)$$

When $w < -1$, the power to which u is raised in (14) is positive, so the integral diverges. However, as previously noted, in this case the universe self-destructs at time t_{rip}, so the integral bounds should really only run up to t_{rip} rather than to infinity! This corresponds to $u = 0$, so it turns out that the upper bound again vanishes and we again get (15). However, now $3w+3$ is negative, so this represents a decreasing rather than increasing function. Inserting absolute values to make this clear, and including (5) for good measure, gives the full story:

$$d_e(t) = \begin{cases} \dfrac{c}{|3w+1|}\left[S^{-1/2} + |3w+3|t\right] & \text{if } w > -1 \\ \dfrac{c}{2}S^{-1/2} & \text{if } w = -1 \\ \dfrac{c}{|3w+1|}\left[S^{-1/2} - |3w+3|t\right] & \text{if } w < -1 \end{cases} \qquad (16)$$